$$h_2 + \frac{V_2^2}{2} = h_1 + \frac{V_1^2}{2} + Q \tag{14.54}$$

This is energy equation, which is valid even in the presence of friction or non-equilibrium conditions between Secs 1 and 2. It is evident that the sum of enthalpy and kinetic energy remains constant in an adiabatic flow. Enthalpy performs a similar role that internal energy performs in a nonflowing system. The difference between the two types of systems is the flow work pv required to push the fluid through a section.

Bernoulli and Euler Equations For inviscid flows, the steady form of the momentum equation is the Euler equation,

$$\frac{dp}{\rho} + V\,dV = 0 \tag{14.55}$$

Integrating along a streamline, we get the Bernoulli's euqation for a compressible flow as

$$\int \frac{dp}{\rho} + \frac{V^2}{2} = \text{constant} \tag{14.56}$$

For adiabatic frictionless flows the Bernoulli's equation is identical to the energy equation. To appreciate this, we have to remember that this is an isentropic flow, so that the $T\,ds$ equation is given by

$$T\,ds = dh - v\,dp$$

which yields

$$dh = \frac{dp}{\rho}$$

Then the Euler equation (14.55) can also be written as

$$V\,dV + dh = 0$$

Needless to say that this is identical to the adiabatic form of the energy Eq. (14.54). The merger of the momentum and energy equation is attributed to the elimination of one of the flow variables due to constant entropy.

Momentum Principle for a Control Volume For a finite control volume between Sections 1 and 2 (Fig. 14.3), the momentum principle is

$$p_1 A_1 - p_2 A_2 + F = \dot{m} V_2 - \dot{m} V_1$$

or

$$p_1 A_1 - p_2 A_2 + F = \rho_2 V_2^2 A_2 - \rho_1 V_1^2 A_1 \tag{14.57}$$

where F is the x component of resultant force exerted on the fluid by the walls. The momentum principle, Eq. (14.57), is applicable even when there are frictional dissipative processes within the control volume.

14.6 STAGNATION AND SONIC PROPERTIES

The stagnation values are useful reference conditions in a compressible flow. Suppose the properties of a flow (such as T, p, ρ, etc.) are known at a point. The stagnation properties at a point are defined as those which are to be obtained if the

local flow were imagined to cease to zero velocity isentropically. The stagnation values are denoted by a subscript zero. Thus, the stangation enthalpy is defined as

$$h_0 = h + \frac{1}{2}V^2$$

For a perfect gas, this yields,

$$c_p T_0 = c_p T + \frac{1}{2}V^2 \tag{14.58}$$

which defines the stagnation temperature. It is meaningful to express the ratio of (T_0/T) in the form

$$\frac{T_0}{T} = 1 + \frac{V^2}{2 c_p T} = 1 + \frac{\gamma - 1}{2} \cdot \frac{V^2}{\gamma RT}$$

or

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} \text{Ma}^2 \tag{14.59}$$

If we know the local temperature (T) and Mach number (Ma), we can find out the stagnation temperature T_0. Consequently, isentropic relations can be used to obtain stagnation pressure and stagnation density as

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma - 1}} = \left[1 + \frac{\gamma - 1}{2}\text{Ma}^2\right]^{\frac{\gamma}{\gamma - 1}} \tag{14.60}$$

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{\gamma - 1}} = \left[1 + \frac{\gamma - 1}{2}\text{Ma}^2\right]^{\frac{1}{\gamma - 1}} \tag{14.61}$$

In general, the stagnation properties can vary throughout the flow field. However, if the flow is adiabatic, then $h + \dfrac{V^2}{2}$ is constant throughout the flow (Eq. 14.54). It follows that the h_0, T_0, and a_0 are constant throughout an adiabatic flow, even in the presence of friction. It is understood that all stagnation properties are constant along an isentropic flow. If such a flow starts from a large reservoir where the fluid is practically at rest, then the properties in the reservoir are equal to the stagnation properties everywhere in the flow (Fig. 14.4).
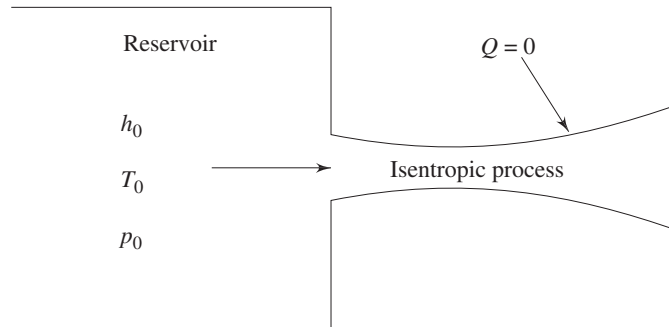

Fig. 14.4 An isentropic process starting from a reservoir

There is another set of conditions of comparable usefulness where the flow is sonic, Ma = 1.0. These sonic, or critical properties are denoted by asterisks: p^*, ρ^*, a^*, and T^*. These properties are attained if the local fluid is imagined to expand or compress isentropically until it reachers Ma = 1.

We have already discussed that the total enthalpy, hence T_0, is conserved so long the process is adiabatic, irrespective of frictional effects. In contrast, the stagnation pressure p_0 and density ρ_0 decrease if there is friction. From Eq. (14.58), we note that

$$V^2 = 2 c_p (T_0 - T)$$

or

$$V = \left[\frac{2\gamma R}{\gamma - 1} (T_0 - T) \right]^{\frac{1}{2}} \tag{14.62a}$$

is the relationship between the fluid velocity and local temperature (T), in an adiabatic flow. The flow can attain a maximum velocity of

$$V_{\max} = \left[\frac{2\gamma R T_0}{\gamma - 1} \right]^{\frac{1}{2}} \tag{14.62b}$$

As it has already been stated, the unity Mach number, Ma = 1, condition is of special significance in compressible flow, and we can now write from Eq. (14.59), (14.60) and (14.61),

$$\frac{T_0}{T^*} = \frac{1 + \gamma}{2} \tag{14.63a}$$

$$\frac{p_0}{p^*} = \left(\frac{1 + \gamma}{2} \right)^{\frac{\gamma}{\gamma - 1}} \tag{14.63b}$$

$$\frac{\rho_0}{\rho^*} = \left(\frac{1 + \gamma}{2} \right)^{\frac{1}{\gamma - 1}} \tag{14.63c}$$

For diatomic gases, like air $\gamma = 1.4$, the numerical values are

$$\frac{T^*}{T_0} = 0.8333, \quad \frac{p^*}{p_0} = 0.5282, \quad \text{and} \quad \frac{\rho^*}{\rho_0} = 0.6339$$

The fluid velocity and acoustic speed are equal at sonic condition and is

$$V^* = a^* = [\gamma R T^*]^{1/2} \tag{14.64a}$$

or

$$V^* = \left[\frac{2\gamma}{\gamma + 1} R T_0 \right]^{\frac{1}{2}} \tag{14.64b}$$

We shall employ both stagnation conditions and critical conditions as reference conditions in a variety of one dimensional compressible flows.

14.6.1 Effect of Area Variation on Flow Properties in Isentropic Flow

In considering the effect of area variation on flow properties in isentropic flow, we shall concern ourselves primarily with the velocity and pressure. We shall determine the effect of change in area, A, on the velocity V, and the pressure p.

From Eq. (14.55), we can write

$$\frac{dp}{\rho} + d\left(\frac{V^2}{2}\right) = 0$$

or $\qquad\qquad dp = -\rho V\, dV$

Dividing by ρV^2, we obtain

$$\frac{dp}{\rho V^2} = -\frac{dV}{V} \qquad\qquad (14.65)$$

A convenient differential form of the continuity equation can be obtained from Eq. (14.50) as

$$\frac{dA}{A} = -\frac{dV}{V} - \frac{d\rho}{\rho}$$

Substituting from Eq. (14.65),

$$\frac{dA}{A} = \frac{dp}{\rho V^2} - \frac{d\rho}{\rho}$$

or $\qquad\qquad \dfrac{dA}{A} = \dfrac{dp}{\rho V^2}\left[1 - \dfrac{V^2}{dp/d\rho}\right] \qquad\qquad (14.66)$

Invoking the relation (14.47b) for isentropic process in Eq. (14.66), we get

$$\frac{dA}{A} = \frac{dp}{\rho V^2}\left[1 - \frac{V^2}{a^2}\right] = \frac{dp}{\rho V^2}[1 - \text{Ma}^2] \qquad\qquad (14.67)$$

From Eq. (14.67), we see that for Ma < 1 an area change causes a pressure change of the same sign, i.e. positive dA means positive dp for Ma < 1. For Ma > 1, an area change causes a pressure change of opposite sign.

Again, substituting from Eq. (14.65) into Eq. (14.67), we obtain

$$\boxed{\frac{dA}{A} = -\frac{dV}{V}[1 - \text{Ma}^2]} \qquad\qquad (14.68)$$

From Eq. (14.68), we see that Ma < 1 an area change causes a velocity change of opposite sign, i.e. positive dA means negative dV for Ma < 1. For Ma > 1, an area change causes a velocity change of same sign.

These results are summarized in Fig. 14.5, and the relations (14.67) and (14.68) lead to the following important conclusions about compressible flows:

(i) At subsonic speeds (Ma < 1) a decrease in area increases the speed of flow. A subsonic nozzle should have a convergent profile and a subsonic diffuser should possess a divergent profile. The flow behaviour in the regime of Ma < 1 is therefore qualitatively the same as in incompressible flows.

(ii) In supersonic flows (Ma > 1), the effect of area changes are different. According to Eq. (14.68), a supersonic nozzle must be built with an increasing area in the flow direction. A supersonic diffuser must be a converging channel. Divergent nozzles are used to produce supersonic flow in missiles and launch vehicles.
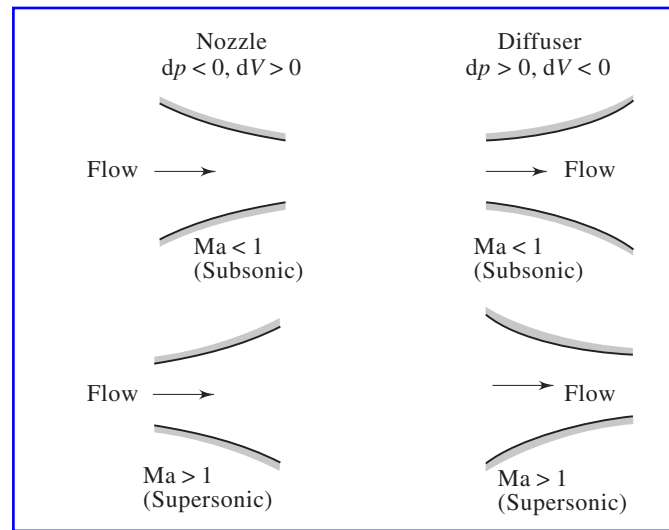
Fig. 14.5   Shapes of nozzles and diffusers in subsonic and supersonic regimes

Suppose a nozzle is used to obtain a supersonic stream staring from low speeds at the inlet (Fig. 14.6). Then the Mach number should increase from Ma = 0 near the inlet to Ma > 1 at the exit. It is clear that the nozzle must converge in the subsonic portion and diverge in the supersonic portion. Such a nozzle is called a *convergent–divergent nozzle*. A convergent–divergent nozzle is also called a *de Laval nozzle*, after Carl G.P. de Laval who first used such a configuration in his steam turbines in late nineteenth century. From Fig. 14.6 it is clear that the Mach number must be unity at the throat, where the area is neither increasing nor decreasing. *This is consistent with Eq. (14.68) which shows that dV can be non-zero at the throat only if Ma = 1.* It also follows that the sonic velocity can be achieved only at the throat of a nozzle or a diffuser.
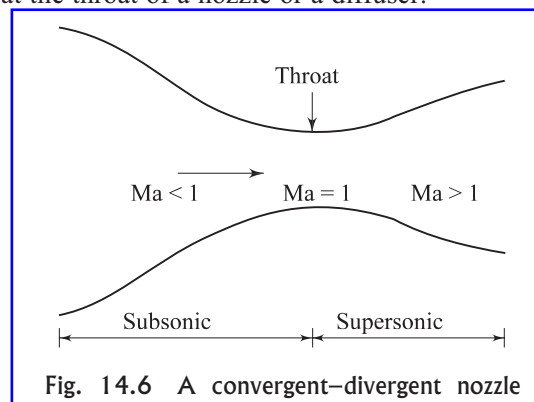

Fig. 14.6   A convergent–divergent nozzle

The condition, however, does not restrict that Ma must necessarily be unity at the throat. According to Eq. (14.68), a situation is possible where Ma ≠ 1 at the throat if $dV = 0$ there. For an example, the flow in a convergent-divergent duct may be subsonic everywhere with Ma increasing in the convergent portion and

decreasing in the divergent portion with Ma ≠ 1 at the throat (see Fig. 14.7). The first part of the duct is acting as a nozzle, whereas the second part is acting as a diffuser. Alternatively, we may have a convergent-divergent duct in which the flow is supersonic everywhere with Ma decreasing in the convergent part and increasing in the divergent part and again Ma ≠ 1 at the throat (see Fig. 14.8).

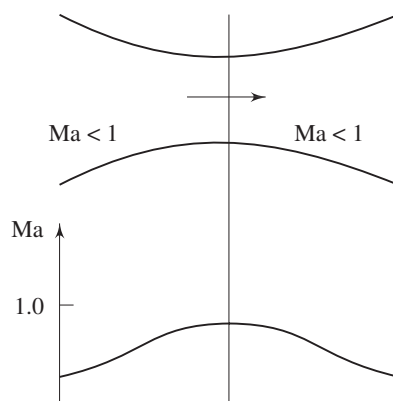**Fig. 14.7**   Convergent-divergent duct with Ma ≠ 1 at throat

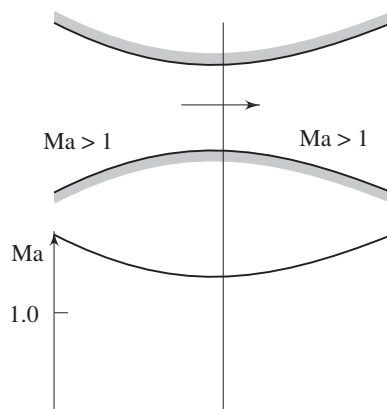**Fig. 14.8**   Convergent-divergent duct with Ma ≠ 1 at throat

14.6.2   Isentropic Flow in a Converging Nozzle

Let us consider the mass flow rate of an ideal gas through a converging nozzle. If the flow is isentropic, we can write

$$\dot{m} = \rho A V$$

or $$\frac{\dot{m}}{A} = \frac{p}{RT} \cdot a\, \text{Ma} \quad \text{[invoking Eqs (14.5) and (14.8)]}$$

or $$\frac{\dot{m}}{A} = \frac{p}{RT} \cdot \sqrt{\gamma\, RT} \cdot \text{Ma}$$

or $$\frac{\dot{m}}{A} = \frac{p}{\sqrt{T}} \cdot \sqrt{\frac{\gamma}{R}} \cdot \text{Ma}$$

or

$$\frac{\dot{m}}{A} = \frac{p}{p_0} \cdot p_0 \cdot \sqrt{\frac{T_0}{T}} \sqrt{\frac{1}{T_0}} \sqrt{\frac{\gamma}{R}} \cdot Ma$$

or

$$\frac{\dot{m}}{A} = \left(\frac{T_0}{T}\right)^{\frac{-\gamma}{\gamma-1}} \cdot \left(\frac{T_0}{T}\right)^{\frac{1}{2}} \frac{p_0}{\sqrt{T_0}} \cdot \sqrt{\frac{\gamma}{R}} \cdot Ma$$

[invoking Eq. (14.44)]

or

$$\frac{\dot{m}}{A} = \sqrt{\frac{\gamma}{R}} \cdot \frac{p_0 Ma}{\sqrt{T_0}} \left(\frac{T_0}{T}\right)^{-\frac{(\gamma+1)}{2(\gamma-1)}}$$

or

$$\frac{\dot{m}}{A} = \sqrt{\frac{\gamma}{R}} \cdot \frac{p_0 Ma}{\sqrt{T_0}} \cdot \frac{1}{\left[1 + \dfrac{\gamma-1}{2} Ma^2\right]^{\frac{(\gamma+1)}{2(\gamma-1)}}}$$ (14.69)

In the expression (14.69), p_0, T_0, γ and R are constant. The discharge per unit area $\dfrac{\dot{m}}{A}$ is a function of Ma only. There exists a particular value of Ma for which $(\dot{m}/A)$ is maximum. Differentiating with respect to Ma and equating it to zero, we get

$$\frac{d(\dot{m}/A)}{dMa} = \sqrt{\frac{\gamma}{R}} \cdot \frac{p_0}{\sqrt{T_0}} \cdot \frac{1}{\left[1 + \dfrac{\gamma-1}{2} Ma^2\right]^{\frac{(\gamma+1)}{2(\gamma-1)}}} + \sqrt{\frac{\gamma}{R}} \cdot \frac{p_0 Ma}{\sqrt{T_0}}$$

$$\left[-\frac{(\gamma+1)}{2(\gamma-1)} \left\{1 + \frac{\gamma-1}{2} Ma^2\right\}^{\frac{-(\gamma+1)}{2(\gamma-1)}-1} \left\{\frac{\gamma-1}{2} 2Ma\right\}\right] = 0$$

or

$$1 - \frac{Ma^2 (\gamma+1)}{2\left\{1 + \dfrac{\gamma-1}{2} Ma^2\right\}} = 0$$

or $Ma^2 (\gamma+ 1) = 2 + (\gamma- 1) Ma^2$

or $Ma = 1$

So, discharge is maximum when   Ma = 1.

We know that $V = aMa = \sqrt{\gamma RT}$ Ma. By logarithmic differentiation, we get

$$\frac{dV}{V} = \frac{dMa}{Ma} + \frac{1}{2} \frac{dT}{T}$$ (14.70)

We also know that

$$\frac{T}{T_0} = \left[1 + \frac{\gamma-1}{2} Ma^2\right]^{-1}$$ (14.59 repeated)

By logarithmic differentiation, we get

$$\frac{dT}{T} = -\frac{(\gamma-1) Ma^2}{1 + \dfrac{(\gamma-1)}{2} Ma^2} \cdot \frac{dMa}{Ma}$$ (14.71)

From Eqs (14.70) and (14.71), we get

$$\frac{\mathrm{d}V}{V} = \frac{\mathrm{d}Ma}{Ma}\left[1 - \frac{\{(\gamma-1)/2\}Ma^2}{1 + \frac{(\gamma-1)}{2}Ma^2}\right]$$

$$\frac{\mathrm{d}V}{V} = \frac{1}{1 + \frac{(\gamma-1)}{2}Ma^2} \cdot \frac{\mathrm{d}Ma}{Ma} \tag{14.72}$$

From Eqs (14.68) and (14.72) we get

$$\frac{\mathrm{d}A}{A}\frac{1}{(Ma^2-1)} = \frac{1}{1 + \frac{\gamma-1}{2}Ma^2} \cdot \frac{\mathrm{d}Ma}{Ma}$$

$$\boxed{\frac{\mathrm{d}A}{A} = \frac{(Ma^2-1)}{1 + \frac{(\gamma-1)}{2}} \cdot \frac{\mathrm{d}Ma}{Ma}} \tag{14.73}$$

By substituting $Ma = 1$ in Eq. (14.73), we get $\mathrm{d}A = 0$ or $A = $ constant. Some $Ma = 1$ can occur only at the throat and nowhere else, and this happens only when the discharge is maximum. When $Ma = 1$, the discharge is maximum and the nozzle is said to be choked. The properties at the throat are termed as critical properties which are already expressed through Eq. (14.63a), (14.63b) and (14.63c). By substituting $Ma = 1$ in Eq. (14.69), we get

$$\frac{\dot{m}}{A^*} = \sqrt{\frac{\gamma}{R}} \cdot \frac{p_0}{\sqrt{T_0}} \cdot \frac{1}{\left[\frac{(\gamma+1)}{2}\right]^{\frac{\gamma+1}{2(\gamma-1)}}} \tag{14.74}$$

(as we have earlier designated critical or sonic conditions by a superscript asterisk). Dividing Eq. (14.74) by Eq. (14.69) we obtain

$$\boxed{\frac{A}{A^*} = \frac{1}{Ma}\left[\left\{\frac{2}{\gamma+1}\right\}\left\{1 + \frac{(\gamma-1)}{2}Ma^2\right\}\right]^{\frac{\gamma+1}{2(\gamma-1)}}} \tag{14.75}$$

From Eq. (14.75) we see that a choice of Ma gives a unique value of A/A^*. The variation of A/A^* with Ma is shown in Fig. 14.9. Note that the curve is double valued; that is, for a given value of A/A^* (other than unity), there are two possible values of Mach number. This signifies the fact that the supersonic nozzle is diverging.
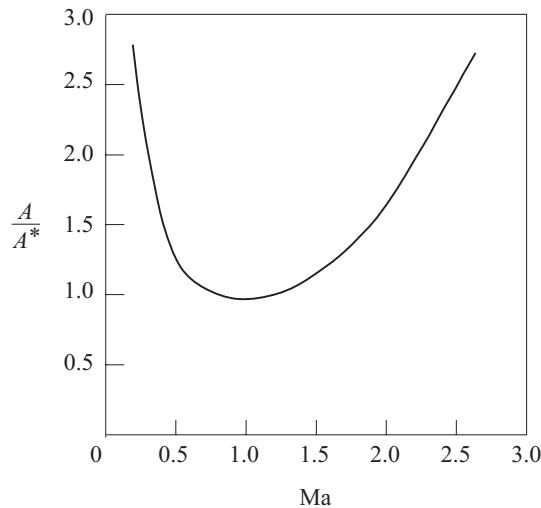
Fig.14.9   Variation of A/A* with Ma in isentropic flow for $Y_g = 1.4$

14.6.3   Pressure Distribution and Choking in a Converging Nozzle

Let us first consider a convergent nozzle as shown in Fig. 14.10(a). Figure 14.10(b) shows the pressure ratio p/p_0 along the length of the nozzle. The inlet conditions of the gas are at the stagnation state (p_0, T_0) which are constants. The pressure at the exit plane of the nozzle is denoted by p_E and the back pressure is p_B which can be varied by the adjustment of the valve. At the condition $p_0 = p_E = p_B$, there shall be no flow through the nozzle. The pressure is p_0 throughout, as shown by condition (i) in Fig. 14.10(b). As p_B is gradually reduced, the flow rate shall increase. The pressure will decrease in the direction of flow as shown by condition (ii) in Fig. 14.10(b). The exit plane pressure p_E shall remain equal to p_B so long as the maximum discharge condition is not reached. Condition (iii) in Fig. 14.10(b) illustrates the pressure distribution in the maximum discharge situation. When ($\dot{m}/A$) attains its maximum value, given by substituting Ma $= 1$ in Eq. (14.69), p_E is equal to p^*. Since the nozzle does not have a diverging section, further reduction in back pressure p_B will not accelerate the flow to supersonic condition. As a result, the exit pressure p_E shall continue to remain at p^* even though p_B is lowered further. The convergent-nozzle discharge against the variation of back pressure is shown in Fig. 14.11. As it has been pointed out earlier, the maximum value of ($\dot{m}/A$) at Ma $= 1$ is stated as the choked flow. With a given nozzle, the flow rate cannot be increased further. thus neither the nozzle exit pressure, nor the mass flow rate are affected by lowering p_B below p^*.
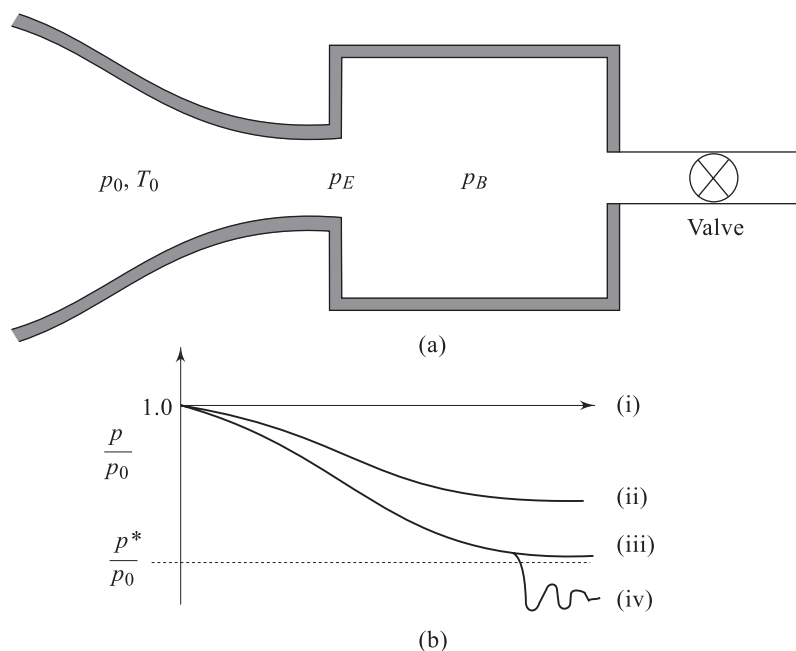
(a)

(b)

Fig. 14.10 (a) Compressible flow through a converging nozzle (b) Pressure distribution along a converging nozzle for different values of back pressure
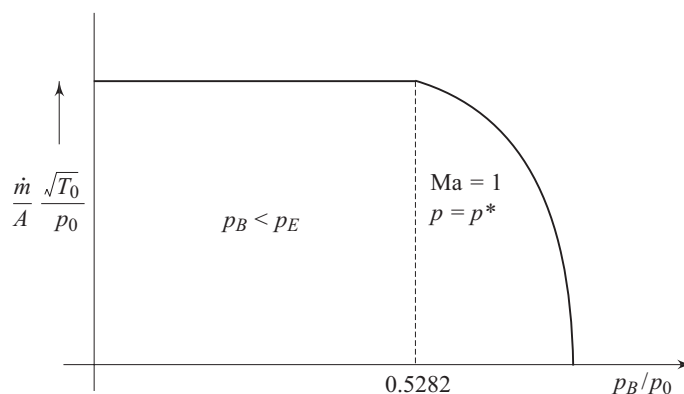

Fig. 14.11 Mass flow rate and the variation of back pressure in a converging nozzle

However for p_B less than p^*, the flow leaving the nozzle has to expand to match the lower back pressure as shown by condition (iv) in Fig. 14.10(b). This expansion process is three-dimensional and the pressure distribution cannot be predicted by one-dimensional theory. Experiments reveal that a series of shocks form in the exit stream, resulting in an increase in entropy.

14.6.4 Isentropic Flow in a Converging-Diverging Nozzle

Now consider the flow in a convergent-divergent nozzle (Fig. 14.12). The upstream stagnation conditions are assumed constant; the pressure in the exit plane of the nozzle is denoted by p_E; the nozzle discharges to the back pressure, p_B. With the value initially closed, there is no flow through the nozzle; the pressure is constant at p_0. Opening the valve slightly produces the pressure distribution shown by curve (i). Completely subsonic flow is discerned. Then p_B is lowered in such a way that sonic condition is reached at the throat (ii). The flow rate becomes maximum for a given nozzle and the stagnation conditions. On further reduction of the back pressure, the flow upstream of the throat does not respond. However, if the back pressure is reduced further (cases (iii) and (iv)), the flow initially becomes supersonic in the diverging section, but then adjusts to the back pressure by means of a normal shock standing inside the nozzle. In such cases, the position of the shock moves downstream as p_B is decreased, and for curve (iv) the normal shock stands right at the exit plane. The flow in the entire divergent portion up to the exit plane is now supersonic. When the back pressure is reduced even further (v), there is no normal shock anywhere within the nozzle, and the jet pressure adjusts to p_B by means of oblique shock waves outside the exit plane. A converging diverging nozzle is generally intended to produce supersonic flow near the exit plane. If the back pressure is set at (vi), the flow will be isentropic throughout the nozzle, and supersonic at nozzle exit. Nozzles operating at $p_B = p_{VI}$ (corresponding to curve (vi) in Fig. 14.12) are said to be at design conditions. Rocket-propelled vehicles use converging-diverging nozzles to accelerate the exhaust gases to the maximum possible velocity to produce high thrust.
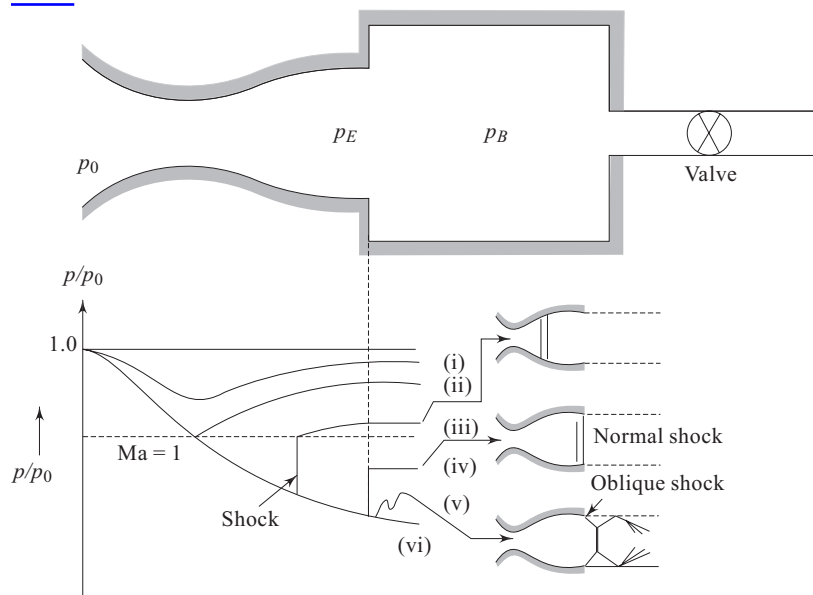

Fig. 14.12   Pressure distribution along a converging-diverging nozzle for different values of back pressure p_B

14.7 NORMAL SHOCKS

Shock waves are highly localized irreversibilities in the flow. Within the distance of a mean free path, the flow passes from a supersonic to a subsonic state, the velocity decreases suddenly and the pressure rises sharply. To be more specific, a shock is said to have occurred if there is an abrupt reduction of velocity in the downstream in course of a supersonic flow in a passage or around a body. Normal shocks are substantially perpendicular to the flow and oblique shocks are inclined at other angles. Shock formation is possible for confined flows as well as for external flows. Normal shock and oblique shock may mutually interact to make another shock pattern. Different type of shocks are presented in Fig. 14.13.
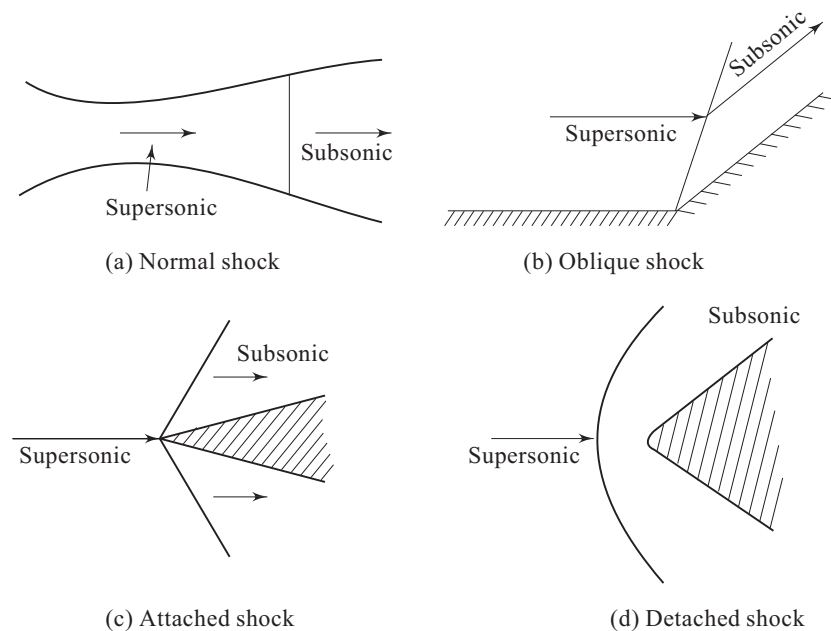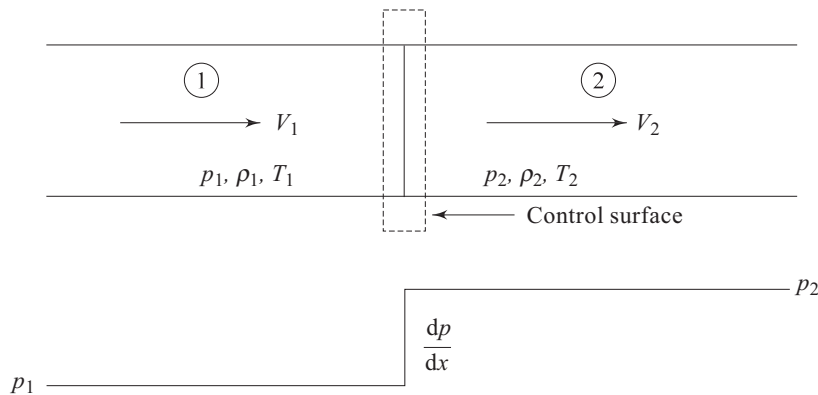

Fig. 14.13   Different type of shocks

Fig. 14.14   One dimensional normal shock

Figure 14.14 shows a control surface that includes a normal shock. The fluid is assumed to be in thermodynamic equilibrium upstream and downstream of the shock, the properties of which are designated by the subscripts 1 and 2, respectively.

Continuity equation can be written as

$$\frac{\dot{m}}{A} = \rho_1 V_1 = \rho_2 V_2 = G \qquad (14.76)$$

where G is the mass velocity kg/m^{2}s.

From momentum equation, one can write

$$p_1 - p_2 = \frac{\dot{m}}{A} (V_2 - V_1) = \rho_2 V_2^2 - \rho_1 V_2^2$$

or $\qquad\qquad p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2 \qquad (14.77a)$

or $\qquad\qquad\qquad F_1 = F_2 \qquad (14.77b)$

where $F = p + \rho V^2$ can be termed as *impulse function*.

The energy equation may be written as

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} = h_{01} = h_{02} = h_0 \qquad (14.78)$$

where h_0 is stagnation enthalpy.

From the second law of thermodynamics, it may be written as

$$s_2 - s_1 \geq 0 \qquad (14.79)$$

But Eq. (14.79) is of little help in calculating actual entropy change across the shock. To calculate the entropy change, we have

$$T\mathrm{d}s = \mathrm{d}h - v\mathrm{d}p \qquad (14.33 \text{ repeated})$$

For an ideal gas we can write

$$\mathrm{d}s = c_p \frac{\mathrm{d}T}{T} - R\frac{\mathrm{d}p}{p}$$

For constant specific heat, this equation can be integrated to give

$$s_2 - s_1 = cp \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} \qquad (14.80)$$

For an ideal gas the equation of state can be written as

$$p = \rho RT \qquad (14.8 \text{ repeated})$$

Equations (14.76), (14.77a), (14.78), (14.80) and (14.8) are the governing equations for the flow of an ideal gas through normal shock. If all the properties at state "1" (upstream of the shock) are known, then we have six unknowns ($T_2, p_2, \rho_2, V_2, h_2, s_2$) in these five equations. However, we have known relationship between h and T [Eq. (14.17)] for an ideal gas which is given by $\mathrm{d}h = c_p \mathrm{d}T$. For an ideal gas with constant specific heats,

$$\Delta h = h_2 - h_1 = c_p(T_2 - T_1) \qquad (14.81)$$

Thus, we have the situation of six equations and six unknowns.

If all the conditions at state "1" (immediately upstream of the shock) are known, how many possible states "2" (immediate downstream of the shock) are there? The mathematical answer indicates that there is a unique state "2" for a given state "1". Before describing the physical picture and precise location of these two states let us introduce Fanno line and Rayleigh line flows.

14.7.1   Fanno Line Flows

If we consider a problem of frictional adiabatic flow through a duct, the governing Eqs (14.76), (14.78), (14.80) (14.8) and (14.81) are valid between any two points "1" and "2". Equation (14.77a) requires to be modified in order to take into account the frictional force, R_x, of the duct wall on the flow and we obtain

$$R_x + p_1 A - p_2 A = \dot{m} V_2 - \dot{m} V_1 \qquad (14.82)$$

So, for a frictional flow, we thus have the situation of six equations and seven unknowns. If all the conditions of "1" are known, how many possible states "2" are there? Mathematically, we get number of possible states "2". With an infinite number of possible states "2" for a given state "1", what do we observe if all possible states "2" are plotted on a *T-s* diagram? The locus of all possible states "2" reachable from state "1" is a continuous curve passing through state "1". However, the question is how to determine this curve? Perhaps the simplest way is to assume different values of T_2. For an assumed value of T_2, the corresponding values of all other properties at "2" and R_x can be determined.
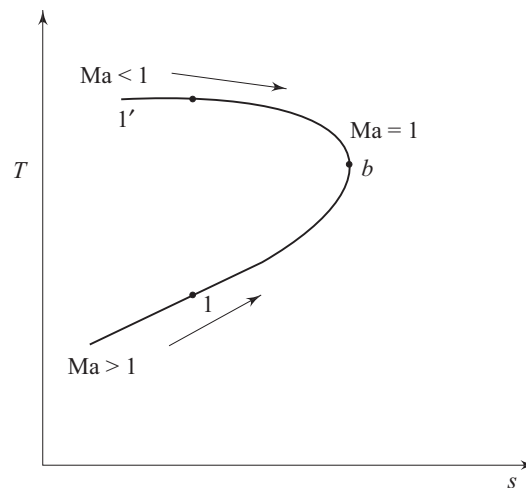

Fig. 14.15   Fanno line representation of constant area adiabatic flow

The locus of all possible downstream states is called Fanno line and is shown in Fig. 14.15. Point "*b*" corresponds to maximum entropy where the flow is sonic. This point splits the Fanno line into subsonic (upper) and supersonic (lower)

portions. If the inlet flow is supersonic and corresponds to point 1 in Fig. 14.15, then friction causes the downstream flow to move closer to point "*b*" with a consequent decrease of Mach number towards unity. Each point on the curve between point 1 and "*b*" corresponds to a certain duct length *L*. As *L* is made larger, the conditions at the exit move closer to point "*b*". Finally, for a certain value of *L*, the flow becomes sonic. Any further increase in *L* is not possible without a drastic revision of the inlet conditions. Consider the alternative case where the inlet flow is subsonic, say, given the point 1′ in Fig. 14.15. As *L* increases, the exit conditions move closer to point "*b*". If *L* is increased to a sufficiently large value, then point "*b*" is reached and the flow at the exit becomes sonic. The flow is again choked and any further increase in *L* is not possible without an adjustment of the inlet conditions.
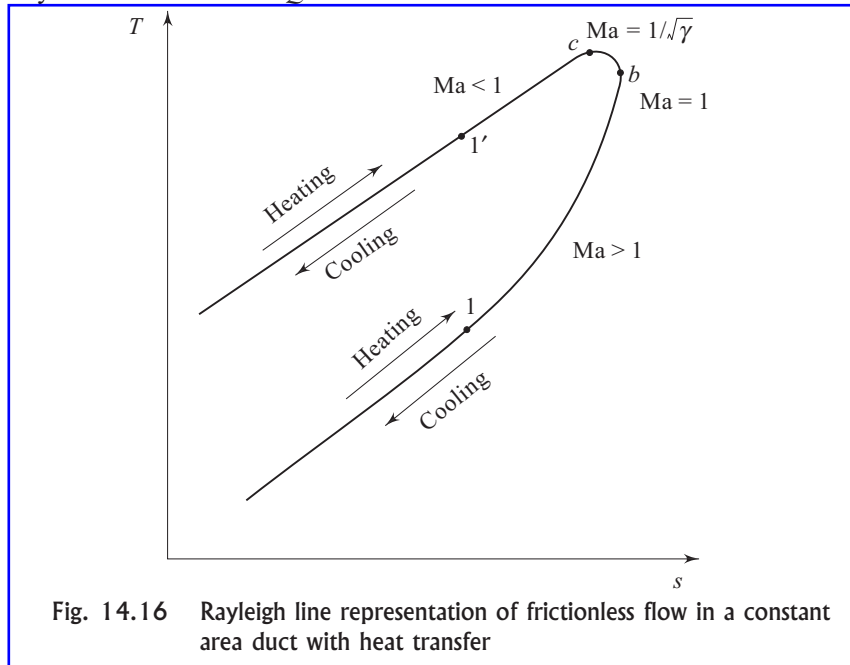
14.7.2 Rayleigh Line Flows

If we consider the effects of heat transfer on a frictionless compressible flow through a duct, the governing Eq. (14.76), (14.77a), (14.80), (14.8) and (14.81) are valid between any two points "1" and "2". Equation (14.78) requires to be modified in order to account for the heat transferred to the flowing fluid per unit mass, d*Q*, and we obtain

$$dQ = h_{02} - h_{01} \tag{14.83}$$

So, for frictionless flow of an ideal gas in a constant area duct with heat transfer, we have again a situation of six equations and seven unknowns. If all conditions at state "1" are known, how many possible states "2" are there? Mathematically, there exists infinite number of possible states "2". With an infinite number of possible states "2" for a given state "1", what do we observe if all possible states "2" are plotted on a *T-s* diagram? The locus of all possible states "2" reachable from state "1" is a continuous curve passing through state "1". Again, the question arises as to how to determine this curve? The simplest way to go about this problem is to assume different values of T_2. For an assumed value of T_2, the corresponding values of all other properties at "2" and δQ can be determined. The results of these calculations are shown on the *T-s* plane in Fig. 14.16. The curve in Fig. 14.16 is called the Rayleigh line.

At the point of maximum temperature (point "*c*" in Fig. 14.16), the value of Mach number for an ideal gas is $1/\sqrt{\gamma}$. At the point of maximum entropy, the Mach number is unity. On the upper branch of the curve, the flow is always subsonic and it increases monotonically as we proceed to the right along the curve. At every point on the lower branch of the curve, the flow is supersonic, and it decreases monotonically as we move to the right along the curve. Irrespective of the initial Mach number, with heat addition, the flow state proceeds to the right and with heat rejection, the flow state proceeds to the left along the Rayleigh line. For example, let us consider a flow which is at an initial state given by 1 on the Rayleigh line in fig. 14.16. If heat is added to the flow, the conditions in the downstream region 2 will move close to point "*b*". The velocity reduces due to increase in pressure and density, and Ma approaches unity. If δQ is increased to a

sufficiently high value, then point "*b*" will be reached and flow in region 2 will be sonic. The flow is again choked, and any further increase in δQ is not possible without an adjustment of the initial condition. The flow cannot become subsonic by any further increase in δQ.

Fig. 14.16   Rayleigh line representation of frictionless flow in a constant area duct with heat transfer

14.7.3   The Physical Picture of the Flow through a Normal Shock

It is possible to obtain physical picture of the flow through a normal shock by employing some of the ideas of Fanno line and Rayleigh line Flows. Flow through a normal shock must satisfy Eqs (14.76), (14.77a), (14.78), (14.80), (14.8) and 14.81). Since all the condition of state "1" are known, there is no difficulty in locating state "1" on *T-s* diagram. In order to draw a Fanno line curve through state "1", we require a locus of mathematical states that satisfy Eqs (14.76), (14.78), (14.80), (14.8) and (14.81). The Fanno line curve does not satisfy Eq. (14.77a). A Rayleigh line curve through state "1" gives a locus of mathematical states that satisfy Eqs (14.76), (14.77a), (14.80), (14.8) and (14.81). The Rayleigh line does not satisfy Eq. (14.78). Both the curves on a same T-s diagram are shown in Fig. 14.17. As we have already pointed out, the normal shock should satisfy all the six equations stated above. At the same time, for a given state "1", the end state "2" of the normal shock must lie on both the Fanno line and Rayleigh line passing through state "1." Hence, the intersection of the two lines at state "2" represents the conditions downstream from the shock. In Fig. 14.17, the flow through the shock is indicated as transition from state "1" to state "2". This is also consistent with directional principle indicated by the second law of thermodynamics, i.e. $s_2 > s_1$. From Fig. 14.17, it is also evident that the

flow through a normal shock signifies a change of speed from supersonic to subsonic. Normal shock is possible only in a flow which is initially supersonic.
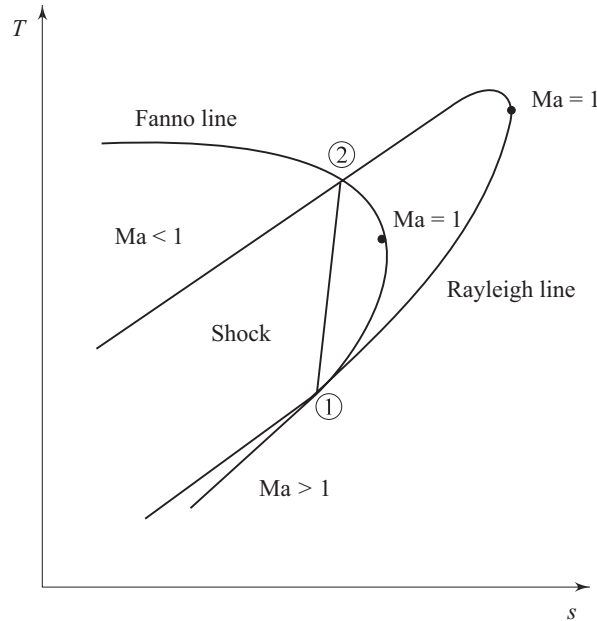

Fig. 14.17 Intersection of Fanno line and Rayleigh line and the solution for normal shock condition

14.7.4 Calculation of Flow Properties Across a Normal Shock

The most easy way to analyze a normal shock is to consider a control surface around the wave as shown in Fig. 14.14. The continuity equation (14.76), the momentum equation (14.77) and the energy equation (14.78) have already been discussed earlier. The energy equation can be simplified for an ideal gas as

$$T_{01} = T_{02} \tag{14.84}$$

By making use of the equation for the speed of sound (14.49) and the equation of state for ideal gas (14.8), the continuity equation can be rewritten to include the influence of Mach number as:

$$\frac{p_1}{RT_1} \text{Ma}_1 \sqrt{\gamma\, RT_1} = \frac{p_2}{RT_2} \text{Ma}_2 \sqrt{\gamma\, RT_2} \tag{14.85}$$

The Mach number can be introduced in momentum equation in the following way:

$$\rho_2 V_2^2 - \rho_1 V_1^2 = p_1 - p_2$$

$$p_1 + \frac{p_1}{RT_1} V_1^2 = p_2 + \frac{p_2}{RT_2} V_2^2$$

$$p_1 (1 + \gamma\, \text{Ma}_1^2) = p_2 (1 + \gamma\, \text{Ma}_2^2) \tag{14.86}$$

Rearranging this equation for the static pressure ratio across the shock wave, we get

$$\frac{p_2}{p_1} = \frac{(1 + \gamma\, \text{Ma}_1^2)}{(1 + \gamma\, \text{Ma}_2^2)} \tag{14.87}$$

As we have already seen that the Mach number of a normal shock wave is always greater than unity in the upstream and less than unity in the downstream, the static pressure always increases across the shock wave.

The energy equation can be written in terms of the temperature and Mach number using the stagnation temperature relationship (14.84) as

$$\frac{T_2}{T_1} = \frac{\{1 + [\gamma - 1)/2]\text{Ma}_1^2}{\{1 + [\gamma - 1)/2]\text{Ma}_2^2} \tag{14.88}$$

Substituting Eqs (14.87) and (14.88) into Eq. (14.85) yields the following relationship for the Mach numbers upstream and downstream of a normal shock wave:

$$\frac{\text{Ma}_1}{1 + \gamma\, \text{Ma}_1^2}\left(1 + \frac{\gamma - 1}{2}\text{Ma}_1^2\right)^{1/2} = \frac{\text{Ma}_2}{1 + \gamma\, \text{Ma}_2^2}\left(1 + \frac{\gamma - 1}{2}\text{Ma}_2^2\right)^{1/2} \tag{14.89}$$

Then, solving this equation for Ma_2 as a function of Ma_1, we obtain two solutions. One solution is trivial, $\text{Ma}_1 = \text{Ma}_2$, which signifies no shock across the control volume. The other solution is

$$\text{Ma}_2^2 = \frac{(\gamma - 1)\,\text{Ma}_1^2 + 2}{2\gamma\,\text{Ma}_1^2 - (\gamma - 1)} \tag{14.90}$$

$\text{Ma}_1 = 1$ in Eq. (14.90) results in $\text{Ma}_2 = 1$. Equations (14.87) and (14.88) also show that there would be no pressure or temperature increase across the shock. In fact, the shock wave corresponding to $\text{Ma}_1 = 1$ is the sound wave across which, by definition, pressure and temperature changes are infinitesimal. Therefore, it can be said that the sound wave represents a degenerated normal shock wave.

14.7.5 Oblique Shock

The discontinuities in supersonic flows do not always exist as normal to the flow direction. There are oblique shocks which are inclined with respect to the flow direction. Let us refer to the shock structure on an obstacle, as depicted qualitatively in Fig.14.18. The segment of the shock immediately in front of the body behaves like a normal shock. Oblique shock is formed as a consequence of the bending of the shock in the free-stream direction. Sometimes in a supersonic flow through a duct, viscous effects cause the shock to be oblique near the walls, the shock being normal only in the core region. The shock is also oblique when a supersonic flow is made to change direction near a sharp corner.
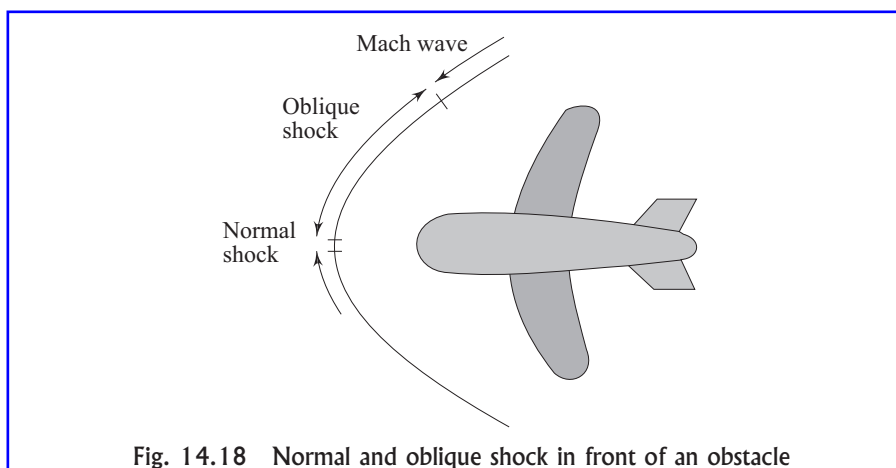
Fig. 14.18 Normal and oblique shock in front of an obstacle

The same relationships derived earlier for the normal shock are valid for the velocity components normal to the oblique shock. The oblique shock continues to bend in the downstream direction until the Mach number of the velocity component normal to the wave is unity. At that instant, the oblique shock degenerates into a so called Mach wave across which changes in flow properties are infinitesimal.

Let us consider a two-dimensional oblique shock as shown in Fig. 14.19.
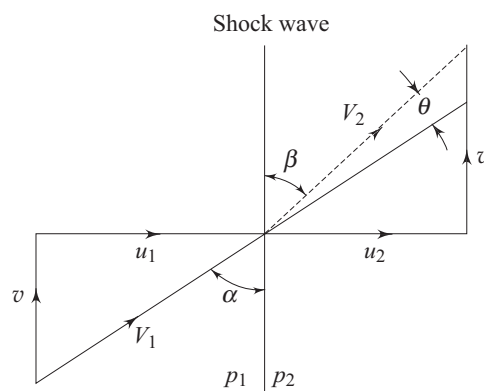

Fig. 14.19 Two dimensional oblique shock

In analyzing flow through such a shock, it may be considered as a normal shock on which a velocity v (parallel to the shock) is superimposed. The change across shock front is determined in the same way as for the normal shock. The equations for mass, momentum and energy conservation are, respectively,

$$\rho_1 u_1 = \rho_2 u_2 \tag{14.91}$$

$$\rho_1 u_1 (u_1 - u_2) = p_2 - p_1 \tag{14.92}$$

$$\frac{\gamma}{\gamma-1} \cdot \frac{p_1}{\rho_1} + \frac{V_1^2}{2} = \frac{\gamma}{\gamma-1} \cdot \frac{p_2}{\rho_2} + \frac{V_2^2}{2}$$

or

$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma-1} \cdot \frac{p_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma-1} \cdot \frac{p_2}{\rho_2} \tag{14.93}$$

These equations are analogous to corresponding equations for normal shock. In addition to these, we have

$$\frac{u_1}{a_1} = \text{Ma}_1 \sin\alpha \text{ and } \frac{u_2}{a_2} = \text{Ma}_2 \sin\beta$$

Then modifying normal shock relations by writing $\text{Ma}_1 \sin\alpha$ and $\text{Ma}_2 \sin\beta$ in place of Ma_1 and Ma_2, we obtain

$$\frac{p_2}{p_1} = \frac{2\gamma \,\text{Ma}_1^2 \sin^2\alpha - \gamma + 1}{\gamma+1} \tag{14.94}$$

$$\frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} = \frac{\tan\beta}{\tan\alpha} = \frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)\,\text{Ma}_1^2 \sin^2\alpha} \tag{14.95}$$

$$\text{Ma}_2^2 \sin^2\beta = \frac{2+(\gamma-1)\,\text{Ma}_1^2 \sin^2\alpha}{1+\tan^2\alpha(\tan\beta/\tan\alpha)} \tag{14.96}$$

Note that although $\text{Ma}_2 \sin\beta < 1$, Ma_2 may be greater than 1. So the flow behind an oblique shock may be supersonic although the normal component of velocity is subsonic. In order to obtain the angle of deflection of flow passing through an oblique shock, we use the relation

$$\tan\theta = \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

$$= \frac{\tan\alpha - (\tan\beta/\tan\alpha)\tan\alpha}{1+\tan^2\alpha(\tan\beta/\tan\alpha)}$$

Having substituted $(\tan\beta/\tan\alpha)$ from Eq. (14.95), finally we get the relation

$$\boxed{\tan\theta = \frac{\text{Ma}_1^2 \sin 2\alpha - 2\cot\alpha}{\text{Ma}_1^2 (\gamma + \cos 2\alpha) + 2}} \tag{14.97}$$

Sometimes, a design is done in such a way that an oblique shock is allowed instead of a normal shock. The losses for the case of oblique shock are much less than those of normal shock. This is the reason for making the nose angle of the fuselage of a supersonic aircraft small.

Summary

- Fluid density varies significantly due to a large Mach number (Ma = V/a) flow. This leads to a situation where continuity and momentum